دانشگاه شهید باهنر کرمان

دانشکده فنی و مهندسی

بخش عمران

سعید شجاعی

علیرضا قربی

وب سایت :

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تحلیل سازه‌ها ۲

اجزاء محدود – توابع پایه

المان میله ۲ گره‌ای

$$EX = \begin{bmatrix} 1 & x \end{bmatrix}$$

$$u(x) = a_1 + a_2 x \quad \rightarrow \quad u(x) = \underbrace{\begin{bmatrix} 1 & x \end{bmatrix}}_{\substack{EX \\ *1}} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

$$\begin{cases} u_1 \rightarrow u(x_s) = a_1 + a_2 x_s \\ u_2 \rightarrow u(x_e) = a_1 + a_2 x_e \end{cases} \rightarrow \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & x_s \\ 1 & x_e \end{bmatrix}}_{X} \times \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = X^{-1} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$X^{-1} = \begin{bmatrix} \dfrac{x_e}{x_e - x_s} & -\dfrac{x_s}{x_e - x_s} \\ -\dfrac{1}{x_e - x_s} & \dfrac{1}{x_e - x_s} \end{bmatrix}$$

$$u(x) = N_1(x)u_1 + N_2(x)u_2 \quad \rightarrow \quad u(x) = \underbrace{\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix}}_{*2} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}}_{*1} = \underbrace{\begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}_{*2}$$

$$EX \left(X^{-1} \times \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \right) = \begin{bmatrix} N_1(x) & N_2(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$N = \begin{bmatrix} EX \end{bmatrix} \times \begin{bmatrix} X^{-1} \end{bmatrix} \quad or \quad \begin{bmatrix} X^{-1} \end{bmatrix}^T \times \begin{bmatrix} EX \end{bmatrix}^T$$

$$B = \frac{dN}{dx}$$

$$K = \iint_A \int_0^L B^T \times E \times B \times dx\,dy\,dz = EA \int_0^L B^T \times B\,dx$$

شکل توابع پایه :

2 Nodes 1D Lagrange Basis Functions

فرم کلی	فرم ساده بین **0** تا **L**
$N = \left[-\dfrac{x - x_e}{x_e - x_s} \quad \dfrac{x - x_s}{x_e - x_s} \right]$	$N = \left[\dfrac{L - x}{L} \quad \dfrac{x}{L} \right]$
$B = \left[-\dfrac{1}{x_e - x_s} \quad \dfrac{1}{x_e - x_s} \right]$	$B = \left[-\dfrac{1}{L} \quad \dfrac{1}{L} \right]$
$K = \begin{bmatrix} \dfrac{AE}{L} & -\dfrac{AE}{L} \\[2mm] -\dfrac{AE}{L} & \dfrac{AE}{L} \end{bmatrix}$	

المان میله ۳ گره‌ای

$$EX = \begin{bmatrix} 1 & x & x^2 \end{bmatrix}$$

$$u(x) = a_1 + a_2 x + a_3 x^2 \quad \rightarrow \quad u(x) = \underbrace{\begin{bmatrix} 1 & x & x^2 \end{bmatrix}}_{EX} \underbrace{\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}}_{*1}$$

$$\begin{cases} u_1 \rightarrow u(x_s) = a_1 + a_2 x_s + a_3 x_s^2 \\ u_2 \rightarrow u\left(\dfrac{x_e}{2} + \dfrac{x_s}{2}\right) = a_1 + a_2 \left(\dfrac{x_e}{2} + \dfrac{x_s}{2}\right) + a_3 \left(\dfrac{x_e}{2} + \dfrac{x_s}{2}\right)^2 \rightarrow \\ u_3 \rightarrow u(x_e) = a_1 + a_2 x_e + a_3 x_e^2 \end{cases}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & x_s & x_s^2 \\ 1 & \dfrac{x_e}{2} + \dfrac{x_s}{2} & \left(\dfrac{x_e}{2} + \dfrac{x_s}{2}\right)^2 \\ 1 & x_e & x_e^2 \end{bmatrix}}_{X} \times \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = X^{-1} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$X^{-1} = \begin{bmatrix} \dfrac{x_e(x_e + x_s)}{(x_e - x_s)^2} & -\dfrac{4 x_e x_s}{(x_e - x_s)^2} & \dfrac{x_s(x_e + x_s)}{(x_e - x_s)^2} \\ -\dfrac{3x_e + x_s}{(x_e - x_s)^2} & \dfrac{4(x_e + x_s)}{(x_e - x_s)^2} & -\dfrac{x_e + 3x_s}{(x_e - x_s)^2} \\ \dfrac{2}{(x_e - x_s)^2} & -\dfrac{4}{(x_e - x_s)^2} & \dfrac{2}{(x_e - x_s)^2} \end{bmatrix}$$
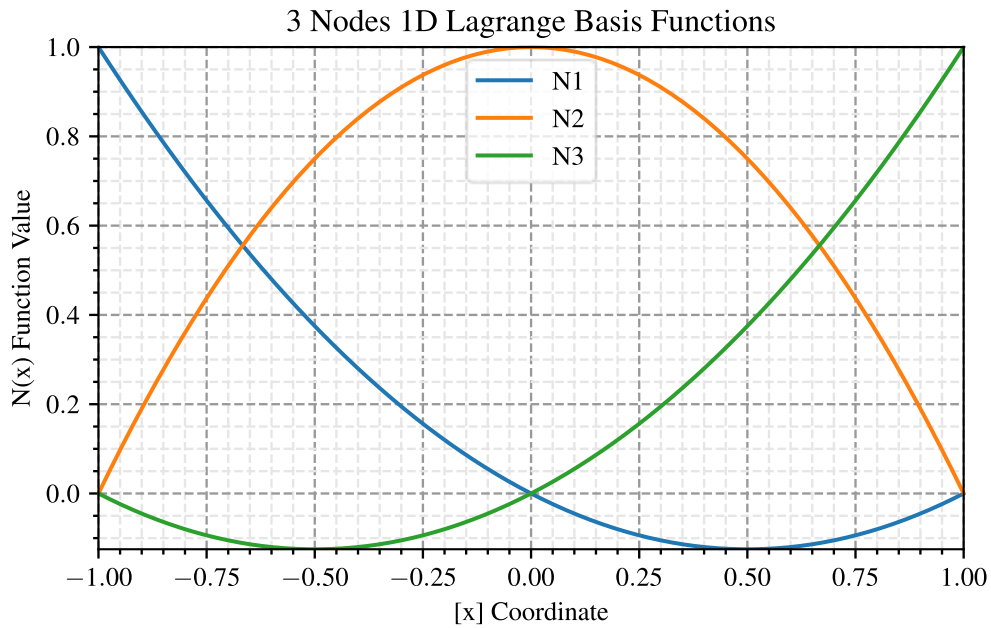
$$u(x) = N_1(x) u_1 + N_2(x) u_2 + N_3(x) u_3 \quad \rightarrow \quad u(x) = \underbrace{\begin{bmatrix} N_1(x) & N_2(x) & N_3(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}}_{*2}$$

$$\underbrace{\underbrace{\begin{bmatrix} 1 & x & x^2 \end{bmatrix}}_{EX} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}}_{*1} = \underbrace{\begin{bmatrix} N_1(x) & N_2(x) & N_3(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}}_{*2}$$

$$EX \left(X^{-1} \times \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \right) = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$N = \begin{bmatrix} EX \end{bmatrix} \times \begin{bmatrix} X^{-1} \end{bmatrix} \quad or \quad \begin{bmatrix} X^{-1} \end{bmatrix}^T \times \begin{bmatrix} EX \end{bmatrix}^T$$

شکل توابع پایه :

3 Nodes 1D Lagrange Basis Functions

فرم کلی	فرم ساده بین **0** تا **L**
$N^T = \begin{bmatrix} -\dfrac{(x-x_e)(x_e-2x+x_s)}{(x_e-x_s)^2} \\[3mm] -\dfrac{4(x-x_e)(x-x_s)}{(x_e-x_s)^2} \\[3mm] -\dfrac{(x-x_s)(x_e-2x+x_s)}{(x_e-x_s)^2} \end{bmatrix}$	$N^T = \begin{bmatrix} 1-\dfrac{3x}{L}+\dfrac{2x^2}{L^2} \\[3mm] \dfrac{4x(L-x)}{L^2} \\[3mm] \dfrac{x(-L+2x)}{L^2} \end{bmatrix}$
$B^T = \begin{bmatrix} -\dfrac{3x_e-4x+x_s}{(x_e-x_s)^2} \\[3mm] \dfrac{4(x_e-2x+x_s)}{(x_e-x_s)^2} \\[3mm] -\dfrac{x_e-4x+3x_s}{(x_e-x_s)^2} \end{bmatrix}$	$B^T = \begin{bmatrix} \dfrac{-3L+4x}{L^2} \\[3mm] \dfrac{4(L-2x)}{L^2} \\[3mm] \dfrac{-L+4x}{L^2} \end{bmatrix}$
$K = \begin{bmatrix} \dfrac{7AE}{3L} & -\dfrac{8AE}{3L} & \dfrac{AE}{3L} \\[3mm] -\dfrac{8AE}{3L} & \dfrac{16AE}{3L} & -\dfrac{8AE}{3L} \\[3mm] \dfrac{AE}{3L} & -\dfrac{8AE}{3L} & \dfrac{7AE}{3L} \end{bmatrix}$	

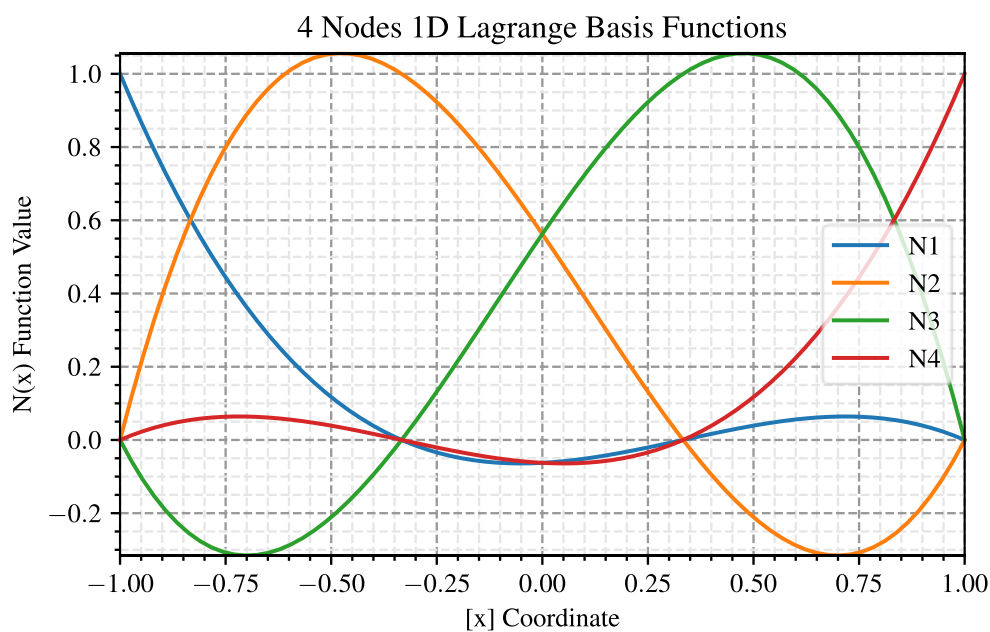# المان میله ۴ گره‌ای

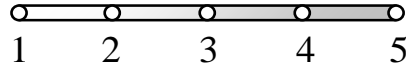$$EX = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & x_s & x_s^2 & x_s^3 \\ 1 & \dfrac{x_e}{3}+\dfrac{2x_s}{3} & \left(\dfrac{x_e}{3}+\dfrac{2x_s}{3}\right)^2 & \left(\dfrac{x_e}{3}+\dfrac{2x_s}{3}\right)^3 \\ 1 & \dfrac{2x_e}{3}+\dfrac{x_s}{3} & \left(\dfrac{2x_e}{3}+\dfrac{x_s}{3}\right)^2 & \left(\dfrac{2x_e}{3}+\dfrac{x_s}{3}\right)^3 \\ 1 & x_e & x_e^2 & x_e^3 \end{bmatrix}$$

شکل توابع پایه :

4 Nodes 1D Lagrange Basis Functions

فرم کلی	فرم ساده بین **0** تا **L**

$$N^T = \begin{bmatrix} -\dfrac{(x-x_e)\left(9x^2 - 9xx_e - 9xx_s + 2x_e^2 + 5x_e x_s + 2x_s^2\right)}{2(x_e - x_s)^3} \\[2em] -\dfrac{9(x-x_e)(x-x_s)(2x_e - 3x + x_s)}{2(x_e - x_s)^3} \\[2em] \dfrac{9(x-x_e)(x-x_s)(x_e - 3x + 2x_s)}{2(x_e - x_s)^3} \\[2em] \dfrac{(x-x_s)\left(9x^2 - 9xx_e - 9xx_s + 2x_e^2 + 5x_e x_s + 2x_s^2\right)}{2(x_e - x_s)^3} \end{bmatrix}$$

$$N^T = \begin{bmatrix} 1 - \dfrac{11x}{2L} + \dfrac{9x^2}{L^2} - \dfrac{9x^3}{2L^3} \\[1.5em] \dfrac{9x\left(2L^2 - 5Lx + 3x^2\right)}{2L^3} \\[1.5em] \dfrac{9x\left(-L^2 + 4Lx - 3x^2\right)}{2L^3} \\[1.5em] \dfrac{x\left(2L^2 - 9Lx + 9x^2\right)}{2L^3} \end{bmatrix}$$

$$B^T = \begin{bmatrix} -\dfrac{27x^2 - 36xx_e - 18xx_s + 11x_e^2 + 14x_e x_s + 2x_s^2}{2(x_e - x_s)^3} \\[2em] \dfrac{9\left(9x^2 - 10xx_e - 8xx_s + 2x_e^2 + 6x_e x_s + x_s^2\right)}{2(x_e - x_s)^3} \\[2em] -\dfrac{9\left(9x^2 - 8xx_e - 10xx_s + x_e^2 + 6x_e x_s + 2x_s^2\right)}{2(x_e - x_s)^3} \\[2em] \dfrac{27x^2 - 18xx_e - 36xx_s + 2x_e^2 + 14x_e x_s + 11x_s^2}{2(x_e - x_s)^3} \end{bmatrix}$$

$$B^T = \begin{bmatrix} -\dfrac{11L^2 - 36Lx + 27x^2}{2L^3} \\[1.5em] \dfrac{18L^2 - 90Lx + 81x^2}{2L^3} \\[1.5em] -\dfrac{9L^2 - 72Lx + 81x^2}{2L^3} \\[1.5em] \dfrac{2L^2 - 18Lx + 27x^2}{2L^3} \end{bmatrix}$$

$$K = \begin{bmatrix} \dfrac{37AE}{10L} & -\dfrac{189AE}{40L} & \dfrac{27AE}{20L} & -\dfrac{13AE}{40L} \\[1.5em] -\dfrac{189AE}{40L} & \dfrac{54AE}{5L} & -\dfrac{297AE}{40L} & \dfrac{27AE}{20L} \\[1.5em] \dfrac{27AE}{20L} & -\dfrac{297AE}{40L} & \dfrac{54AE}{5L} & -\dfrac{189AE}{40L} \\[1.5em] -\dfrac{13AE}{40L} & \dfrac{27AE}{20L} & -\dfrac{189AE}{40L} & \dfrac{37AE}{10L} \end{bmatrix}$$
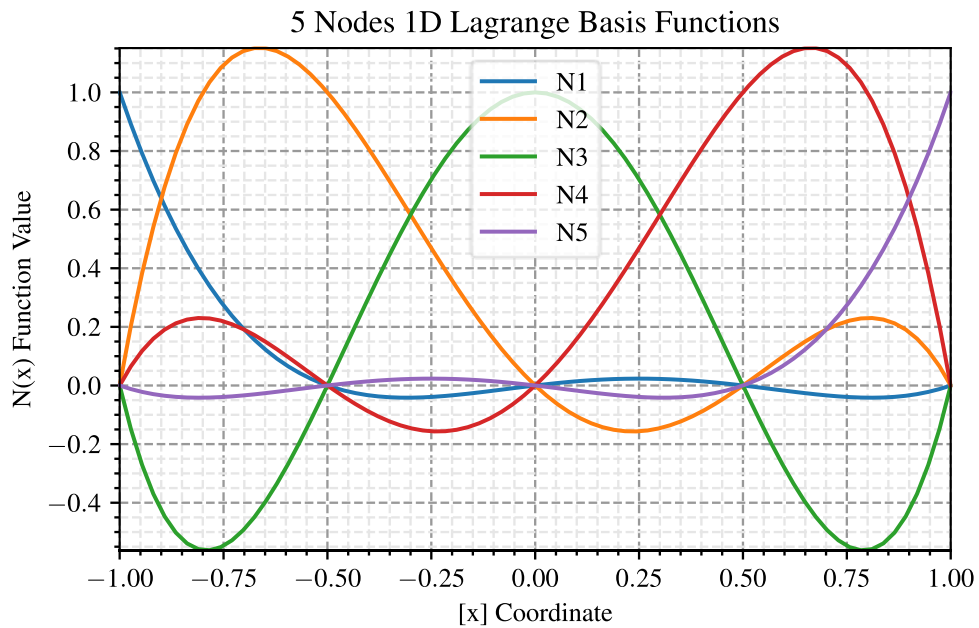
المان میله ۵ گره‌ای

$$EX = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & x_s & x_s^2 & x_s^3 & x_s^4 \\ 1 & \dfrac{x_e}{4} + \dfrac{3x_s}{4} & \left(\dfrac{x_e}{4} + \dfrac{3x_s}{4}\right)^2 & \left(\dfrac{x_e}{4} + \dfrac{3x_s}{4}\right)^3 & \left(\dfrac{x_e}{4} + \dfrac{3x_s}{4}\right)^4 \\ 1 & \dfrac{x_e}{2} + \dfrac{x_s}{2} & \left(\dfrac{x_e}{2} + \dfrac{x_s}{2}\right)^2 & \left(\dfrac{x_e}{2} + \dfrac{x_s}{2}\right)^3 & \left(\dfrac{x_e}{2} + \dfrac{x_s}{2}\right)^4 \\ 1 & \dfrac{3x_e}{4} + \dfrac{x_s}{4} & \left(\dfrac{3x_e}{4} + \dfrac{x_s}{4}\right)^2 & \left(\dfrac{3x_e}{4} + \dfrac{x_s}{4}\right)^3 & \left(\dfrac{3x_e}{4} + \dfrac{x_s}{4}\right)^4 \\ 1 & x_e & x_e^2 & x_e^3 & x_e^4 \end{bmatrix}$$

شکل توابع پایه :

5 Nodes 1D Lagrange Basis Functions

فرم کلی :

$$N^T = \begin{bmatrix} -\dfrac{(x-x_e)\left(-32x^3+48x^2x_e+48x^2x_s-22xx_e^2-52xx_ex_s-22xx_s^2+3x_e^3+13x_e^2x_s+13x_ex_s^2+3x_s^3\right)}{3(x_e-x_s)^4} \\[1em] -\dfrac{16(x-x_e)(x-x_s)\left(8x^2-10xx_e-6xx_s+3x_e^2+4x_ex_s+x_s^2\right)}{3(x_e-x_s)^4} \\[1em] \dfrac{4(x-x_e)(x-x_s)\left(16x^2-16xx_e-16xx_s+3x_e^2+10x_ex_s+3x_s^2\right)}{(x_e-x_s)^4} \\[1em] -\dfrac{16(x-x_e)(x-x_s)\left(8x^2-6xx_e-10xx_s+x_e^2+4x_ex_s+3x_s^2\right)}{3(x_e-x_s)^4} \\[1em] -\dfrac{(x-x_s)\left(-32x^3+48x^2x_e+48x^2x_s-22xx_e^2-52xx_ex_s-22xx_s^2+3x_e^3+13x_e^2x_s+13x_ex_s^2+3x_s^3\right)}{3(x_e-x_s)^4} \end{bmatrix}$$

$$B^T = \begin{bmatrix} -\dfrac{-128x^3+240x^2x_e+144x^2x_s-140xx_e^2-200xx_ex_s-44xx_s^2+25x_e^3+65x_e^2x_s+35x_ex_s^2+3x_s^3}{3(x_e-x_s)^4} \\[1em] \dfrac{16\left(-32x^3+54x^2x_e+42x^2x_s-26xx_e^2-56xx_ex_s-14xx_s^2+3x_e^3+17x_e^2x_s+11x_ex_s^2+x_s^3\right)}{3(x_e-x_s)^4} \\[1em] -\dfrac{4\left(-64x^3+96x^2x_e+96x^2x_s-38xx_e^2-116xx_ex_s-38xx_s^2+3x_e^3+29x_e^2x_s+29x_ex_s^2+3x_s^3\right)}{(x_e-x_s)^4} \\[1em] \dfrac{16\left(-32x^3+42x^2x_e+54x^2x_s-14xx_e^2-56xx_ex_s-26xx_s^2+x_e^3+11x_e^2x_s+17x_ex_s^2+3x_s^3\right)}{3(x_e-x_s)^4} \\[1em] -\dfrac{-128x^3+144x^2x_e+240x^2x_s-44xx_e^2-200xx_ex_s-140xx_s^2+3x_e^3+35x_e^2x_s+65x_ex_s^2+25x_s^3}{3(x_e-x_s)^4} \end{bmatrix}$$

فرم ساده بین **0** و **L** :

$$N^T = \begin{bmatrix} \dfrac{3L^4 - 25L^3x + 70L^2x^2 - 80Lx^3 + 32x^4}{3L^4} \\[3mm] \dfrac{16x(3L^3 - 13L^2x + 18Lx^2 - 8x^3)}{3L^4} \\[3mm] \dfrac{4x(-3L^3 + 19L^2x - 32Lx^2 + 16x^3)}{L^4} \\[3mm] \dfrac{16x(L^3 - 7L^2x + 14Lx^2 - 8x^3)}{3L^4} \\[3mm] \dfrac{x(-3L^3 + 22L^2x - 48Lx^2 + 32x^3)}{3L^4} \end{bmatrix}$$

$$B^T = \begin{bmatrix} \dfrac{-25L^3 + 140L^2x - 240Lx^2 + 128x^3}{3L^4} \\[3mm] -\dfrac{-48L^3 + 416L^2x - 864Lx^2 + 512x^3}{3L^4} \\[3mm] \dfrac{-12L^3 + 152L^2x - 384Lx^2 + 256x^3}{L^4} \\[3mm] -\dfrac{-16L^3 + 224L^2x - 672Lx^2 + 512x^3}{3L^4} \\[3mm] \dfrac{-3L^3 + 44L^2x - 144Lx^2 + 128x^3}{3L^4} \end{bmatrix}$$

$$K = \begin{bmatrix} \dfrac{985AE}{189L} & -\dfrac{6848AE}{945L} & \dfrac{1016AE}{315L} & -\dfrac{1472AE}{945L} & \dfrac{347AE}{945L} \\[3mm] -\dfrac{6848AE}{945L} & \dfrac{3328AE}{189L} & -\dfrac{4736AE}{315L} & \dfrac{5888AE}{945L} & -\dfrac{1472AE}{945L} \\[3mm] \dfrac{1016AE}{315L} & -\dfrac{4736AE}{315L} & \dfrac{496AE}{21L} & -\dfrac{4736AE}{315L} & \dfrac{1016AE}{315L} \\[3mm] -\dfrac{1472AE}{945L} & \dfrac{5888AE}{945L} & -\dfrac{4736AE}{315L} & \dfrac{3328AE}{189L} & -\dfrac{6848AE}{945L} \\[3mm] \dfrac{347AE}{945L} & -\dfrac{1472AE}{945L} & \dfrac{1016AE}{315L} & -\dfrac{6848AE}{945L} & \dfrac{985AE}{189L} \end{bmatrix}$$
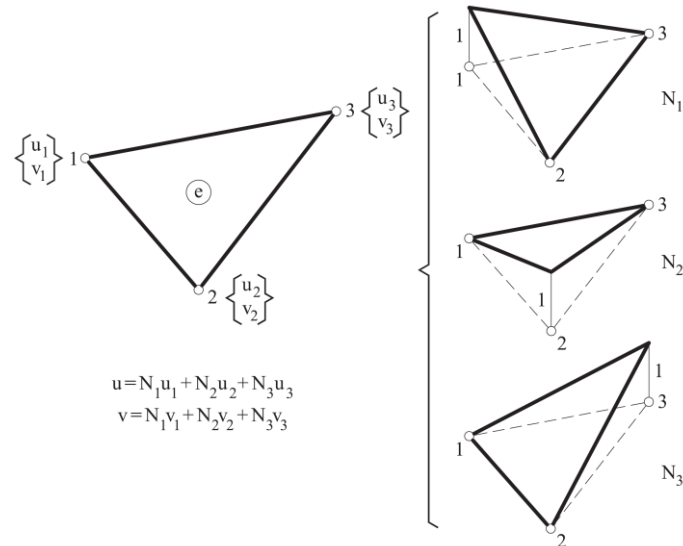
المان مثلث ۳ گره‌ای (درجه ۱)

$$A_{Triangle} = \frac{1}{2}\left(x_1 y_2 - x_2 y_1 - x_1 y_3 + x_3 y_1 + x_2 y_3 - x_3 y_2\right)$$

$$X = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \xrightarrow{\ INVERSE\ } X^{-1} = \begin{pmatrix} \dfrac{x_2 y_3 - x_3 y_2}{2A} & -\dfrac{x_1 y_3 - x_3 y_1}{2A} & \dfrac{x_1 y_2 - x_2 y_1}{2A} \\ \dfrac{y_2 - y_3}{2A} & -\dfrac{y_1 - y_3}{2A} & \dfrac{y_1 - y_2}{2A} \\ -\dfrac{x_2 - x_3}{2A} & \dfrac{x_1 - x_3}{2A} & -\dfrac{x_1 - x_2}{2A} \end{pmatrix}$$

$$XI = \frac{1}{2A}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \xrightarrow{\ TRANSPOSE\ } \frac{1}{2A}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} \times \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

$$\begin{pmatrix} N_1 = \dfrac{C_{11} + C_{21}x + C_{31}y}{2A} \\ N_2 = \dfrac{C_{12} + C_{22}x + C_{32}y}{2A} \\ N_3 = \dfrac{C_{13} + C_{23}x + C_{33}y}{2A} \end{pmatrix} \Rightarrow N = \begin{pmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{pmatrix}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$B = \begin{pmatrix} \dfrac{\partial N_1}{\partial x} & 0 & \dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_3}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1}{\partial y} & 0 & \dfrac{\partial N_2}{\partial y} & 0 & \dfrac{\partial N_3}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_3}{\partial x} \end{pmatrix} \Rightarrow B = \frac{1}{2A}\begin{pmatrix} C_{21} & 0 & C_{22} & 0 & C_{23} & 0 \\ 0 & C_{31} & 0 & C_{32} & 0 & C_{33} \\ C_{31} & C_{21} & C_{32} & C_{22} & C_{33} & C_{23} \end{pmatrix}$$
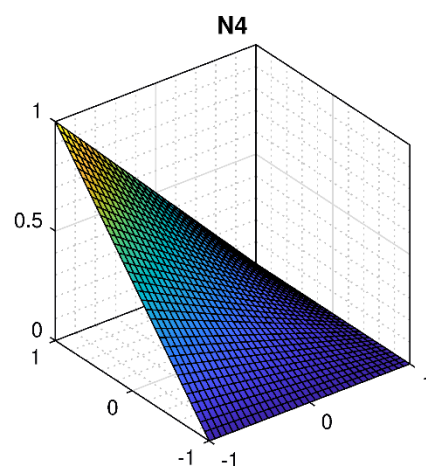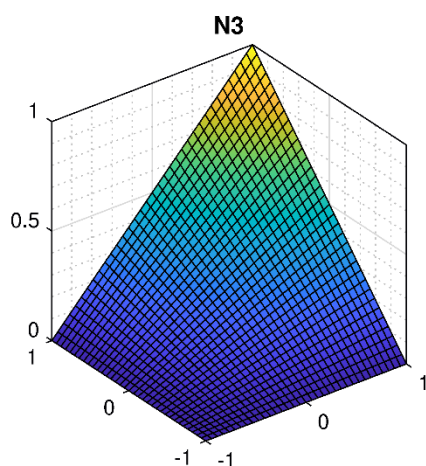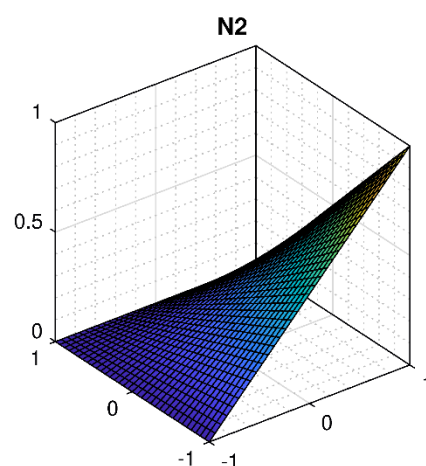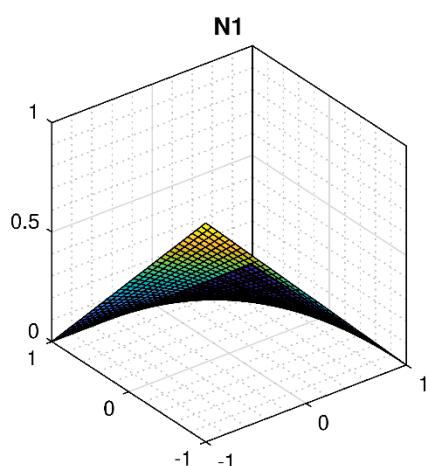
المان چهارگوش منظم ۴ گره‌ای (درجه ۱)

$$X = \begin{bmatrix} 1 & -b & -h & bh \\ 1 & b & -h & -bh \\ 1 & b & h & bh \\ 1 & -b & h & -bh \end{bmatrix} \xrightarrow{INVERSE} X^{-1} = \begin{bmatrix} \dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{4} \\ -\dfrac{1}{4b} & \dfrac{1}{4b} & \dfrac{1}{4b} & -\dfrac{1}{4b} \\ -\dfrac{1}{4h} & -\dfrac{1}{4h} & \dfrac{1}{4h} & \dfrac{1}{4h} \\ \dfrac{1}{4bh} & -\dfrac{1}{4bh} & \dfrac{1}{4bh} & -\dfrac{1}{4bh} \end{bmatrix} \xrightarrow{TRANSPOSE}$$

$$\left(X^{-1}\right)^T = \begin{bmatrix} \dfrac{1}{4} & -\dfrac{1}{4b} & -\dfrac{1}{4h} & \dfrac{1}{4bh} \\ \dfrac{1}{4} & \dfrac{1}{4b} & -\dfrac{1}{4h} & -\dfrac{1}{4bh} \\ \dfrac{1}{4} & \dfrac{1}{4b} & \dfrac{1}{4h} & \dfrac{1}{4bh} \\ \dfrac{1}{4} & -\dfrac{1}{4b} & \dfrac{1}{4h} & -\dfrac{1}{4bh} \end{bmatrix} \xrightarrow{\times EXPANSION} N^T = \begin{bmatrix} \dfrac{1}{4} & -\dfrac{1}{4b} & -\dfrac{1}{4h} & \dfrac{1}{4bh} \\ \dfrac{1}{4} & \dfrac{1}{4b} & -\dfrac{1}{4h} & -\dfrac{1}{4bh} \\ \dfrac{1}{4} & \dfrac{1}{4b} & \dfrac{1}{4h} & \dfrac{1}{4bh} \\ \dfrac{1}{4} & -\dfrac{1}{4b} & \dfrac{1}{4h} & -\dfrac{1}{4bh} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ xy \end{bmatrix}$$

$$\begin{bmatrix} N_1 = \dfrac{(b-x)(h-y)}{4bh} \\ N_2 = \dfrac{(b+x)(h-y)}{4bh} \\ N_3 = \dfrac{(b+x)(h+y)}{4bh} \\ N_4 = \dfrac{(h+y)(b-x)}{4bh} \end{bmatrix} \Rightarrow N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

المان چهارگوش نامنظم ۴ گره‌ای (درجه ۱)

$$X = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{\textit{INVERSE}} X^{-1} = \begin{bmatrix} \dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{4} \\[6pt] -\dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{4} & -\dfrac{1}{4} \\[6pt] -\dfrac{1}{4} & -\dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{4} \\[6pt] \dfrac{1}{4} & -\dfrac{1}{4} & \dfrac{1}{4} & -\dfrac{1}{4} \end{bmatrix} \xrightarrow{\textit{TRANSPOSE}}$$

$$\left(X^{-1}\right)^T = \begin{bmatrix} \dfrac{1}{4} & -\dfrac{1}{4} & -\dfrac{1}{4} & \dfrac{1}{4} \\[6pt] \dfrac{1}{4} & \dfrac{1}{4} & -\dfrac{1}{4} & -\dfrac{1}{4} \\[6pt] \dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{4} \\[6pt] \dfrac{1}{4} & -\dfrac{1}{4} & \dfrac{1}{4} & -\dfrac{1}{4} \end{bmatrix} \xrightarrow{\times \textit{EXPANSION}} N^T = \begin{bmatrix} \dfrac{1}{4} & -\dfrac{1}{4} & -\dfrac{1}{4} & \dfrac{1}{4} \\[6pt] \dfrac{1}{4} & \dfrac{1}{4} & -\dfrac{1}{4} & -\dfrac{1}{4} \\[6pt] \dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{4} & \dfrac{1}{4} \\[6pt] \dfrac{1}{4} & -\dfrac{1}{4} & \dfrac{1}{4} & -\dfrac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ r \\ s \\ rs \end{bmatrix}$$

N1

N2

N3

N4

$$\begin{bmatrix} N_1 = \dfrac{(1-r)(1-s)}{4} \\[2mm] N_2 = \dfrac{(1+r)(1-s)}{4} \\[2mm] N_3 = \dfrac{(1+r)(1+s)}{4} \\[2mm] N_4 = \dfrac{(1-r)(1+s)}{4} \end{bmatrix} \Rightarrow N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$